# ECON5152 — Network Analysis

## Week 9: Diffusion in Networks

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# Outline

# Motivation

## Diffusion

Transmission of a phenomenon across a population.

## Core question

How is transmission *shaped by networks*?

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How is transmission *shaped by networks*?

Examples

- **Spread of diseases through social contacts**
- Diffusion of knowledge and ideas
- Learning through social interactions
- Financial distress through banking networks
- Technology adoption among firms

# Motivation

Diffusion can have many outcomes

- No diffusion takes place
- Diffusion grows in the short run but fades out
- Waves of diffusion across nodes (back and forth exchange)
- Diffusion confined to a subset of the network
- Diffusion reaches all network
- Permanent changes introduced by diffusion (new equilibrium)

# Diffusion Example: Covid-19

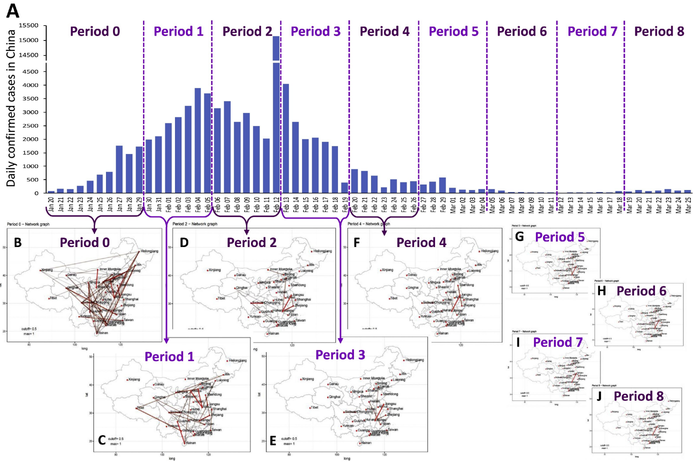


Figure: Daily confirmed cases in China and network graphs (Figure 1 in So et al., July, 2020)

# Diffusion Example: Covid-19

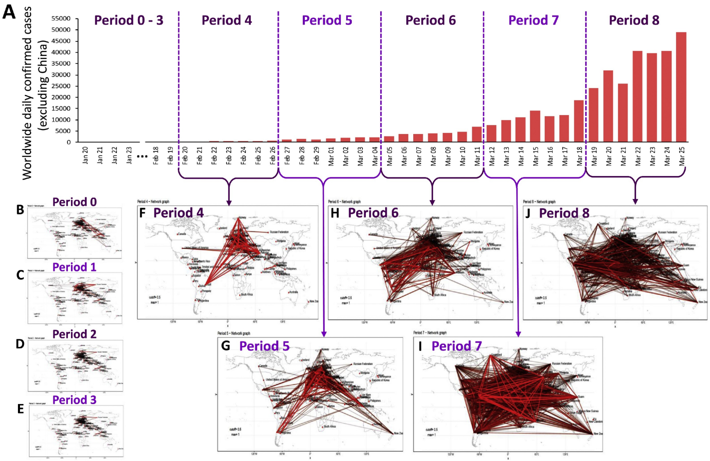


Figure: Worldwide daily confirmed cases outside China and network graphs (Figure 2 in So et al., July, 2020)

# Review: Linear Diffusion

- We have already covered linear diffusion models
- Consider nodes $\mathcal{N} := \{1, \ldots, n\}$
- Let $x_i(t)$ state of node $i \in \mathcal{N}$ at time $t$
- State can refer to: influence, demand, disease status, etc.
- Network linkages captured by adjacency matrix $\mathbf{A}$
- Intuitively: my next state is a baseline plus network average of current state of others (times a peer effect)

$$x_i(t+1) = \alpha + \beta \cdot \sum_j A_{ij} \cdot x_j(t)$$

# Example: Centrality

- Major example of linear diffusion is *Katz-Bonacich centrality*
- Intuitively: node is is important if connected to important nodes
- Importance becomes self-referential
- Let $k_i$ denote Katz-Bonacich centrality of node $i$
- Consider updating influence $x_i(t)$ at time $t$ as

$$x_i(t) = \alpha + \beta \cdot \sum_j A_{ij} \cdot x_j(t)$$

- Centrality is long-run average of influence after linear diffusion

$$k_i = \lim_{t \to \infty} x_i(t)$$

# Connected vs Disconnected Networks

- Diffusion requires connectivity
- Diffusion cannot spread between components
- Component size is strong determinant of diffusion

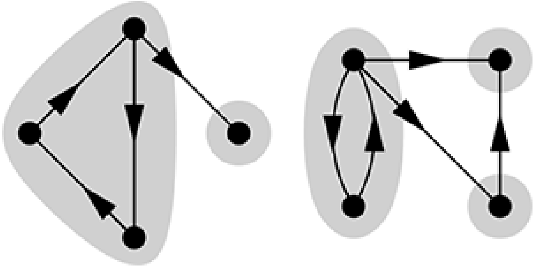

Figure: Shock diffusion in a disconnected network

# Key Takeaways from Linearity

1. Strength of diffusion captured by direct linkages and paths $(\mathbf{A}, \mathbf{A}^2, \mathbf{A}^3, \ldots)$
2. Iterative process examined in the long-run limit
3. Useful to visualize as approximation to large networks (size grows in the limit)
4. Properties of diffusion governed by structure of $\mathbf{A}$
5. Connectedness is key for diffusion
6. These takeaways are relevant in more complex diffusion settings

# Basic Non-linear Diffusion

- Start considering more interesting diffusion processes
- First, abstract away from network altogether
- Assume a complete network (anyone can diffuse to anyone else)
- Covers basic but key models:
  - SI (susceptible-immune) epidemiological model and extensions
  - Bass model of product / service / behaviour adoption

# SI and Bass Models

- Unified framework for simple non-linear diffusion
- $F(t)$ fraction infected / has adopted out of entire population
- $p$ probability of randomly developing disease (fraction of early adopters)
- $q$ infection rate (probability of adoption or imitation)

# SI and Bass Models

- Unified framework for simple non-linear diffusion
- $F(t)$ fraction infected / has adopted out of entire population
- $p$ probability of randomly developing disease (fraction of early adopters)
- $q$ infection rate (probability of adoption or imitation)
- Infection / adoption evolves across time (non-linearly)

$$F(t+1) - F(t) = \underbrace{[p + q \cdot F(t)]}_{\text{early adoption + imitation}} \cdot \underbrace{[1 - F(t)]}_{\text{not yet adopted}}$$

- Intuitively: change in infection / adoption depends on
  1. how many random infections / early adopters remain
  2. infection (learning, imitation) by those not yet infected / adopted

# Solution to SI and Bass Models

- Solution at any time $t$ can be found in closed-form
- Assume no infection / adoption at time $t = 0 \implies F(0) = 0$
- Solution follows a logistic (S-shaped) pattern

$$F(t) = \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} \cdot e^{-(p+q)t}}$$

- Properties of logistic growth
  - slow initial growth
  - explosive diffusion
  - eventual saturation
  - guaranteed spread in the limit: $F(t) \to 1$ as $t \to \infty$

# Logistic Growth

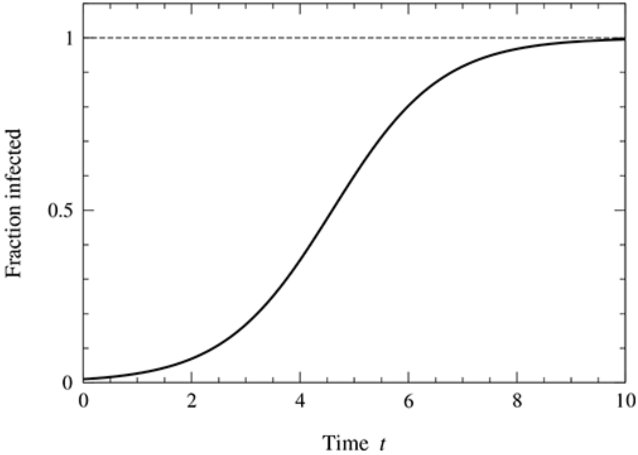


Figure: Logistic (S-shaped) pattern of growth from SI and Bass model

# Motivation: Percolation

- Re-introduce network into diffusion

**Simple diffusion question**

Can something travel from one side of a network to the other?

- Intuitively: is there a path in the network for diffusion to happen?
- Theory that seeks to understand this question: *percolation*
- Origin in statistical physics and materials science
- Example: can water pass through a porous material?

# Translation to Network Diffusion

Using the example of disease spread

- Disease is carried from one node to the others
- Can the disease be carried through the entire population?
- That is, does the disease eventually infect everyone?

# Translation to Network Diffusion

Using the example of disease spread

- Disease is carried from one node to the others
- Can the disease be carried through the entire population?
- That is, does the disease eventually infect everyone?
- From before: answer is *yes* in basic diffusion settings
- We also know disconnection stops diffusion
- What if we are allowed to remove some “immune” nodes?

# Applications of Diffusion Models

Diffusion notion extended to other contexts:

- Financial networks and risk spread when institutions can crash
- Production networks and global supply chains with interruptions (e.g., natural disasters)
- Carrying messages without losing information to "corrupted" nodes (e.g., cryptography)
- Internet connections with router malfunction (estimated $\approx 3\%$ worldwide)
- Criminal networks disrupted by arrests
- Knowledge transmission across generations

# Percolation

- Basic diffusion models show full spread is inevitable with complete networks
- *Percolation theory* studies how this occurs for general networks
- We say that *percolation* can occur if path exists for spread to happen
- Consider how removing nodes affects diffusion
- First consider: what if we keep / remove nodes at random?

1. Probability nodes remain active: $\phi$ (occupation probability)
2. Probability nodes are removed: $\pi \coloneqq 1 - \phi$

# Percolation with Uniform Node Removal

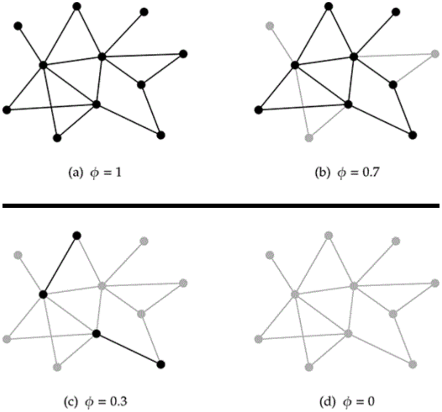


Figure: Networks keeping a fraction $\phi$ of nodes active

Percolation can occur for networks above the line but not those below the line

# Giant Component

## Definition

A *giant component* is a connected component containing a significant fraction of nodes in the network.

- Let $S$ be the fraction of nodes in the giant component (out of $n$)
- Formally, “significant fraction” means $S > 0$ as network size grows $n \to \infty$

# Giant Component

## Definition

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- Let $S$ be the fraction of nodes in the giant component (out of $n$)
- Formally, "significant fraction" means $S > 0$ as network size grows $n \to \infty$
- If $S \approx 0$, no diffusion happens
- As $S \to 1$ we are back in complete case
- Diffusion can only happen if giant component exists

# Percolation Threshold

## Critical probability

The *critical probability* that a giant component exists, defined as $\phi_c$

- Critical probability provides a *percolation threshold*
- Giant component exists (i.e., diffusion is possible) if $\phi > \phi_c$
- Intuitively: as long as we keep $\phi > \phi_c$ nodes active, spread is inevitable
- If $\phi < \phi_c$, not sufficient nodes to maintain diffusion spread

# Threshold for Simple Networks

For most networks, we can find the critical threshold in closed-form

$$\phi_c = \frac{\mathbb{E}[d]}{\mathbb{E}[d^2] - \mathbb{E}[d]}$$

- $\mathbb{E}[d]$ refers to *expected degree*
- $\mathbb{E}[d^2]$ is related to variance of degrees
- Assumes these expected values are the same for all nodes!
- Also assumes network is large: $n \to \infty$

# Threshold for Simple Networks

Simple networks with equal expected degree:

1. Regular degree networks (all same degree $d$)

$$\phi_c = \frac{1}{d-1}$$

2. Erdős–Rényi random graph with probability $p$ or $\mathcal{G}(n, p)$

$$\phi_c = \frac{1}{(n-2)p}$$

3. Configuration model: you choose degree sequence $(d_1, \ldots, d_n) \implies$ fixed $\mathbb{E}[d]$ and $\mathbb{E}[d^2]$

$$\phi_c = \frac{\mathbb{E}[d]}{\mathbb{E}[d^2] - \mathbb{E}[d]}$$

# Threshold for Simple Networks: Theory

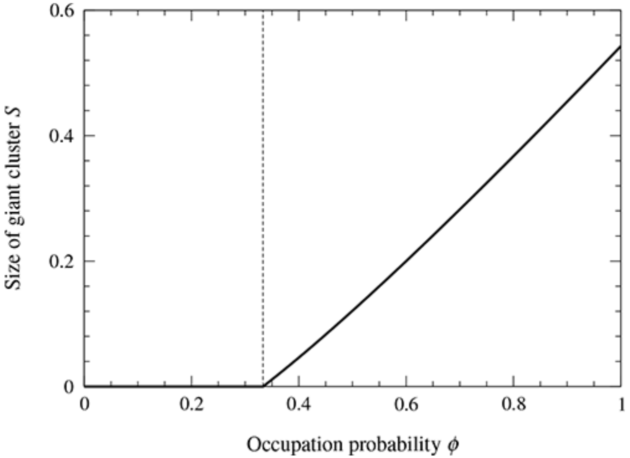



Figure: Size of giant component as function of occupation probability $\phi$ ($n \to \infty$)

# Threshold for Simple Networks: Reality

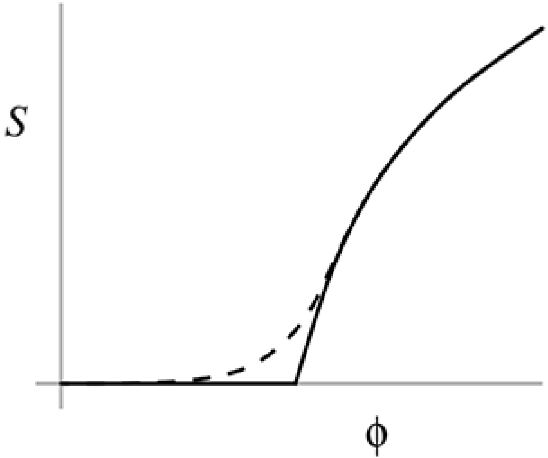



Figure: Size of giant component as function of occupation probability $\phi$ ($n$ fixed)

# Threshold for Power-Law Networks

- Threshold calculation is more complex with different degrees
- Fix ideas and consider power-law degrees with exponent $\alpha$

$$\Pr(d_i = k) = c(\alpha) \cdot k^{-\alpha}$$

- Usually $2 \leq \alpha \leq 3$ in real-world networks (e.g., internet has $\alpha \approx 2.5$)
- For these networks, $\mathbb{E}[d_i]$ exists but $\mathbb{E}[d_i^2] \to \infty$
- This means $\phi_c = 0$ and diffusion is again inevitable
- As all $\phi \geq \phi_c = 0$, any network creates a giant component
- Intuitively: as large degrees are likely, can never fully stop transmission

# Giant Component in Power-law Networks

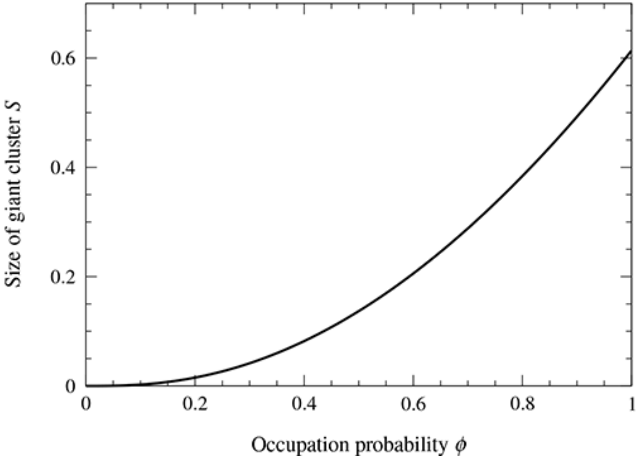



Figure: Size of giant component as function of occupation probability $\phi$ ($\alpha = 2.5$)

# Percolation with Non-uniform Node Removal

- Previous ideas rely on keeping / removing nodes at random
- If aiming to maximize diffusion, better policies exist
- Example: keep / remove nodes whose degrees is largest
  - Vaccinate more likely spreaders
  - Provide information to most connected individual
  - Arrest more central criminal actor
- Introduce probability $\pi_k$ of *removing* nodes with degree equal to $k$
- Example: *remove* all nodes $i$ with degrees larger than or equal to $k_0$

$$\pi_k = \begin{cases} 0 & \text{if } d_i >= k_0 \\ 1 & \text{if } d_i < k_0 \end{cases}$$

# Giant Component with Non-Uniform Removal: Simple Networks

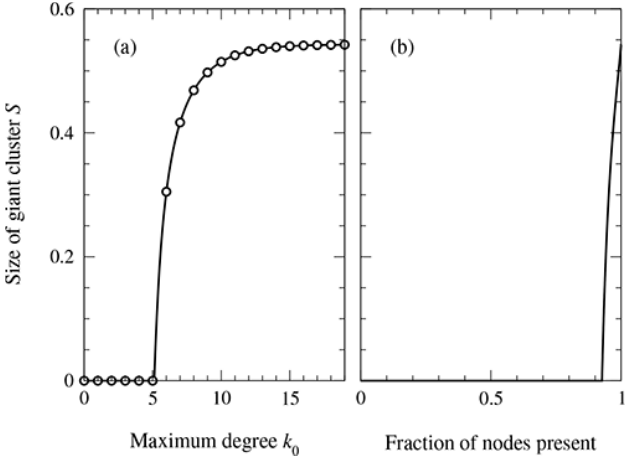



Figure: Size of giant component vs. maximum degree and fractions of nodes kept

# Giant Component with Non-Uniform Removal: Power-Law Networks

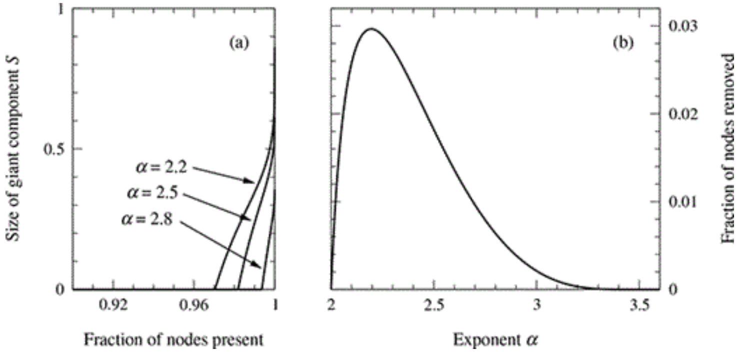



Figure: Size of giant component vs. fractions of nodes kept and power-law exponent

# Extensions

- We can extend original SI and Bass models to include network interaction
- Individuals can only infect / learn from each other if connected
- Percolation consequences are similar than in random networks
- Additional extensions change percolation properties:
  - SIR model: allow individuals to *recover* and become immune
  - SIS model: allow individuals to recover but can become *reinfected*
  - SIRS model: allow individuals to be recover and be immune for a while but potentially *lose immunity*

# Diffusion of Shocks

With diffusion processes in place, consider original motivation:

**Key Question**

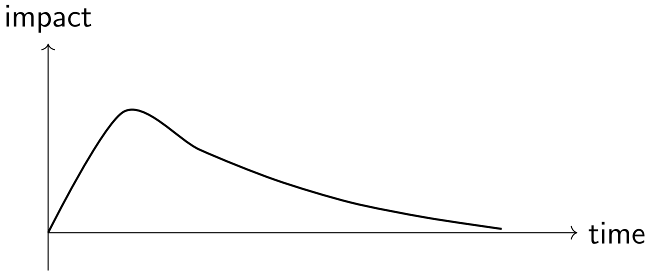
How do shocks spread through a networked population or economy?

Examples:

- spread of financial distress across banks
- spread of panic during bank runs
- rapid adoption of technologies or behaviors
- spread of infectious diseases

Network structure determines whether shocks remain local or propagate widely.

# Linear Diffusion of a Shock




- With linear diffusion systems, a shock propagates through the network
- Initial reaction amplified through linkages
- Transitory shock always gradually returns to baseline

# Nonlinear Diffusion and Cascades

In non-linear diffusion systems:

- small shocks can trigger large reactions
- effects may propagate across many agents
- transitory shocks can be amplified to both systemic and permanent effects
- *percolation in context:* insuring even a large fraction of agents might not guarantee contagion can be controlled

## Cascade

A *cascade* occurs when a small shock spreads widely through the network and generates large system-wide effects.

# Fragility

## Fragility

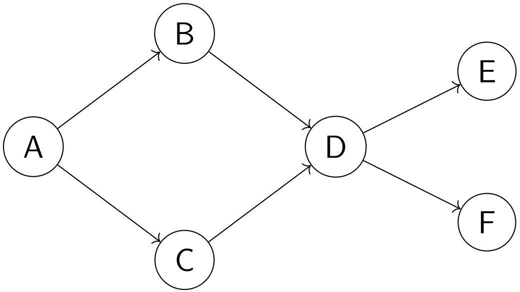
A system is fragile if cascades are likely to occur and can generate large disruptions.

Examples:

- Small infection that spreads quickly can cover a globalized world (Covid)
- Small panic induces bank runs (everyone withdraws money at same time) that creates economic recessions
- Failure of a few institutions triggers systemic financial collapse (e.g. Lehman Brothers and AIG bailout in US)
- Early influential adopters trigger widespread adoption of a product
- Humans massively adopt behaviours that greatly increase survival

# Financial Network Example

Financial systems form complex networks of obligations



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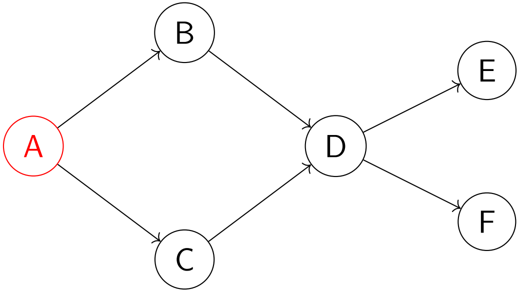
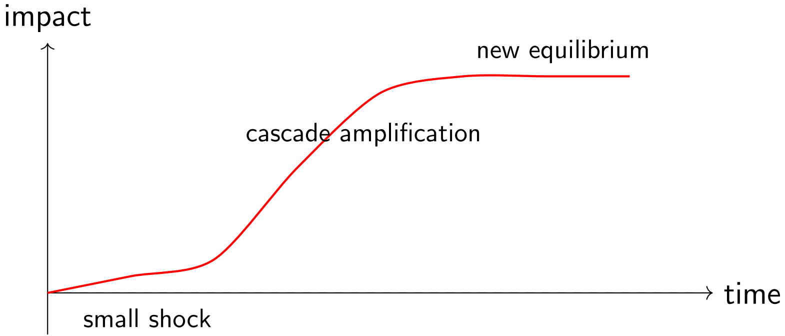



If bank A defaults:

- B and C incur losses
- Weakens D's position indirectly
- Additional failures may follow downstream

# Financial Network Cascade



# Key Takeaways

- Diffusion models describe how states propagate through networks
- Linear diffusion produces predictable dynamics that converge over time
- Basic models such as SI and Bass generate S-shaped diffusion patterns
- Network structure determines whether diffusion reaches the entire population
- Percolation theory explains when large connected components emerge
- Non-linear dynamics can produce cascades and systemic fragility