Appendices

A The Bass model

The main mathematical tools that are necessary for understanding the Bass model come from survival analysis. Bass (1969) defines the probability of buying a product given that no purchase has been made previously, i.e., the hazard function denoted by $H(t)$, as a linear function of the number $Y(t)$ of buyers until time t, the ultimate market potential m, and the parameters of innovation p and imitation q, such that

$$H(t) = \frac{f(t)}{1 - F(t)} = p + \frac{q}{m} Y(t) \,, \tag{A.1}$$

where $f(t)$ represents the likelihood that the product is purchased at any given time, and $1-F(t)$ is the probability that a purchase has not been made before time t. Denoting the total sales at time t by $S(t)$, then $S(t) = mf(t)$, and cumulative sales are given by $Y(t) = mF(t)$. Replacing and re-ordering Eq. (A.1) gives what is known in the mathematics literature as a Riccati equation with constant coefficients; a non-linear differential equation with initial condition $F(0) = 0$, of the form

$$f(t) = p + (q - p)F(t) - q[F(t)]^2 \,.$$

Solving the equation via the separation of variables and evaluating the constant of integration gives the following expressions for $F(t)$, $f(t)$ and $S(t)$:

$$F(t) = \frac{1 - e^{-t(p+q)}}{1 + (q/p)e^{-t(p+q)}} \,, \tag{A.2}$$

$$f(t) = \frac{[(p+q)^2/p]e^{-t(p+q)}}{\left[1 + (q/p)e^{-t(p+q)}\right]^2} \,, \tag{A.3}$$

$$S(t) = \frac{[m(p+q)^2/p]e^{-t(p+q)}}{\left[1 + (q/p)e^{-t(p+q)}\right]^2} \,. \tag{A.4}$$

By maximizing $S(t)$, it is possible to find the moment at which the sales rate reaches its peak, i.e., by differentiating and setting the result equal to zero. The result is found to be

$$t^* = -\frac{\ln(p/q)}{p+q} \,. \tag{A.5}$$

For the generalized Bass model (GBM), the only change necessary is the addition of a multiplicative term on the right side of Eq.(A.1). This is referred to as the market effort, $x(t)$, and is assumed to be (in the original paper of Bass et al., 1994)

$$x(t) = 1 + \beta_1 \frac{P(t) - P(t-1)}{P(t-1)} + \beta_2 \max\left\{0, \frac{A(t) - A(t-1)}{A(t-1)}\right\}.$$

Transforming the market effort into continuous time and integrating between zero and t, one obtains the cumulative market effort, $X(t)$:

$$X(t) = t + \beta_1 \ln\left[\frac{P(t)}{P(0)}\right] + \beta_2 \ln\left[\frac{A(t)}{A(0)}\right] \,.$$

If we assume $X(0) = 0$, then all of the equations remain similar if one replaces t with $X(t)$.

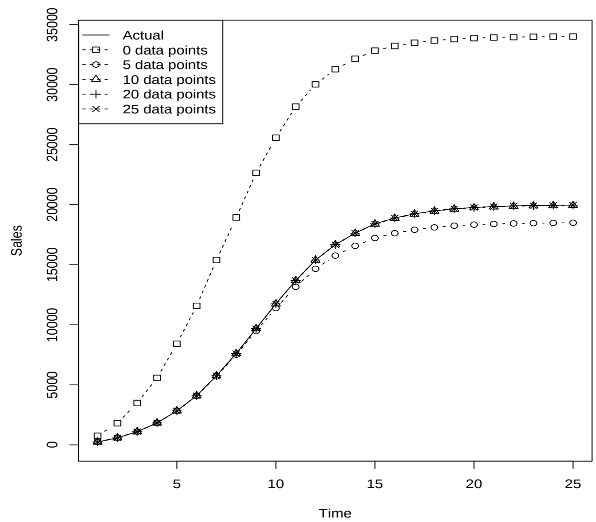
B Simulation results

Table B.1: Bayesian, NLS and ML estimation with different amounts of information.

	$t=0$	$t=5$	$t=8$	$t=9$	$t=10$	$t=15$	$t=20$	$t=25$
Bayesian estimation: informative prior								
p	0.018	0.009	0.010	0.010	0.010	0.010	0.010	0.010
	(0.011)	(0.003)	(3.19×10^{-4})	(7.74×10^{-5})	(3.38×10^{-5})	(1.38×10^{-5})	(1.34×10^{-5})	(1.16×10^{-5})
q	0.410	0.398	0.404	0.401	0.401	0.400	0.400	0.400
	(0.114)	(0.027)	(0.010)	(0.005)	(0.001)	(2.20×10^{-4})	(1.96×10^{-4})	(1.69×10^{-4})
m	32,253.723	25,047.834	19,707.115	19,954.148	19,965.211	19,993.282	19,994.421	19,995.012
	(23,589.898)	(9,813.18)	(936.932)	(307.938)	(58.504)	(7.194)	(6.565)	(5.844)
β_1	−0.812	−1.012	−1.438	−1.490	−1.505	−1.501	−1.497	−1.497
	(0.815)	(0.690)	(0.186)	(0.101)	(0.047)	(0.011)	(0.010)	(0.010)
β_2	1.896	1.094	1.142	1.062	1.023	1.009	1.010	1.009
	(2.173)	(0.886)	(0.362)	(0.191)	(0.097)	(0.014)	(0.013)	(0.012)
β_3	−2.566	−3.149	−2.782	−2.592	−2.539	−2.509	−2.510	−2.510
	(2.520)	(1.254)	(0.654)	(0.279)	(0.135)	(0.021)	(0.020)	(0.018)
Peak	7	10	10	10	10	10	10	10
Take-off	—	6	5	5	5	5	5	5
Saturation	12	14		13	13	13	13	13
Bayesian estimation: non-informative prior								
p	−0.657	—	0.010	0.010	0.010	0.010	0.010	0.010
	(55.392)		(0.001)	(1.00×10^{-4})	(4.23×10^{-5})	(1.33×10^{-5})	(1.32×10^{-5})	(1.18×10^{-5})
q	−3.598	—	0.406	0.401	0.400	0.400	0.400	0.400
	(57.873)		(0.021)	(0.004)	(0.001)	(2.11×10^{-4})	(1.95×10^{-4})	(1.73×10^{-4})
m	1.428	—	19,653.424	19,958.246	19,974.448	19,993.142	19,994.476	19,995.173
	(52.046)		(1,671.902)	(293.729)	(76.331)	(6.983)	(6.679)	(5.908)
β_1	2.091	—	−1.476	−1.505	−1.514	−1.502	−1.497	−1.497
	(79.848)		(0.352)	(0.130)	(0.058)	(0.010)	(0.010)	(0.009)
β_2	0.435	—	1.094	1.029	1.008	1.009	1.010	1.009
	(57.282)		(0.715)	(0.297)	(0.122)	(0.014)	(0.014)	(0.012)
β_3	0.474	—	−2.808	−2.550	−2.519	−2.508	−2.510	−2.510
	(52.968)		(1.425)	(0.437)	(0.170)	(0.020)	(0.021)	(0.018)
Peak	11	—	10	10	10	10	10	10
Take-off	1.5	—	5	5	5	5	5	5
Saturation	—	—	13		13	13	13	13
NLS estimation								
p	—	—	0.010	0.010	0.010	0.010	0.010	0.010
			(4.32×10^{-5})	(1.70×10^{-5})	(1.36×10^{-5})	(1.03×10^{-5})	(1.12×10^{-5})	(1.07×10^{-5})
q	—	—	0.401	0.400	0.401	0.400	0.400	0.400
			(0.001)	(0.001)	(4.83×10^{-4})	(1.64×10^{-4})	(1.65×10^{-4})	(1.57×10^{-4})
m	—	—	19,907.534	19,995.903	19,971.042	19,993.242	19,994.587	19,995.109
			(129.726)	(56.806)	(24.537)	(5.361)	(5.555)	(5.301)
β_1	—	—	−1.506	−1.511	−1.511	−1.502	−1.497	−1.497
			(0.024)	(0.021)	(0.019)	(0.008)	(0.009)	(0.009)
β_2	—	—	1.032	1.018	1.010	1.009	1.010	1.009
			(0.055)	(0.048)	(0.042)	(0.011)	(0.012)	(0.011)
β_3	—	—	−2.596	−2.542	−2.523	−2.508	−2.510	−2.509
			(0.108)	(0.075)	(0.058)	(0.016)	(0.018)	(0.017)
Peak	—	—	10	10	10	10	10	10
Take-off	—	—	5	5	5	5	5	5
Saturation	—	—	13		13	13	13	13
ML estimation								
p	—	0.010	0.010	0.010	0.010	0.010	0.010	0.010
		—	(4.32×10^{-4})	(2.87×10^{-4})	(2.83×10^{-4})	(2.51×10^{-4})	(2.37×10^{-4})	(2.28×10^{-4})
q	—	0.405	0.401	0.400	0.405	0.400	0.400	0.400
		(0.012)	(0.009)	(0.001)	(0.008)	(0.004)	(0.003)	(0.003)
m	—	19873.735	19875.124	19874.812	19874.392	19875.055	19917.957	19991.963
		—	(792.534)	—	(412.169)	(129.526)	(112.097)	(109.329)
β_1	—	−1.270	−1.499	−1.524	−0.912	−1.114	−1.486	−1.488
		(2.512)	(0.456)	(0.430)	(0.436)	(0.290)	(0.262)	(0.261)
β_2	—	0.760	1.046	0.549	2.313	1.862	1.005	0.959
		(1.977)	(1.024)	(0.927)	(0.883)	(0.334)	(0.310)	(0.306)
β_3	—	−2.273	−2.629	2.020	−4.112	−2.204	−2.522	−2.480
		(3.526)	(2.023)	(1.513)	(1.350)	(0.508)	(0.456)	(0.451)
Peak	—	7	8	9	10	10	10	10
Take-off	—	—	5	5	5	5	5	5
Saturation	—	12	11	13	13	13	13	13

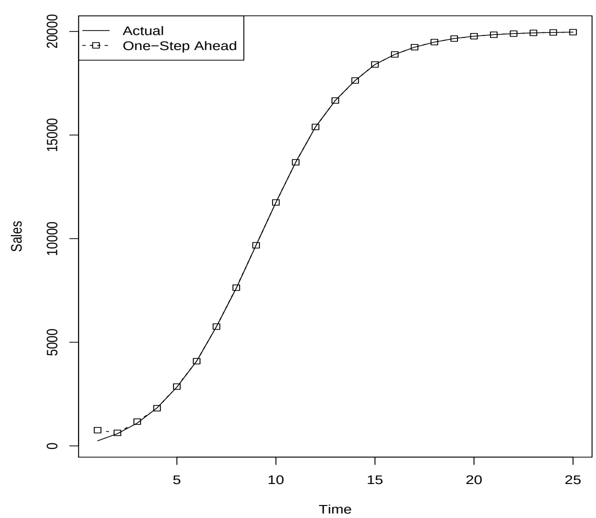
Notes: The models are fitted on each subsample up to t. Informative priors use information from previous markets, while non-informative priors are centered at zero with a high variance. NLS and MLE include appropriate parameter restrictions. Standard errors are given in parentheses (standard deviation of chains for Bayesian methods and asymptotic for frequentist). "—" represents unavailable estimation results.

Figure B.1: Cumulative sales forecasts with information at different time periods.



Notes: Forecasts from Bayesian estimation with informative priors in the simulated data set. Parameter point estimates are used to compute full forecasts.

Figure B.2: One-step-ahead forecasts of cumulative sales with information at different time periods.



Notes: Forecasts from Bayesian estimation with informative priors in the simulated data set. Parameter point estimates are used to compute the one-step-ahead forecasts.

Table B.2: Bayesian, NLS and ML estimation without estimating the short-run market.

	$t=0$	$t=5$	$t=8$	$t=9$	$t=10$	$t=15$	$t=20$	$t=25$
	Bayesian estimation: informative prior							
p	0.018	0.010	0.010	0.010	0.010	0.010	0.010	0.010
	(0.014)	(0.001)	$(1.03{\times}10^{-4})$	$(4.38{\times}10^{-5})$	$(2.57{\times}10^{-5})$	$(1.19{\times}10^{-5})$	$(1.15{\times}10^{-5})$	$(1.03{\times}10^{-5})$
q	0.420	0.401	0.400	0.400	0.400	0.400	0.400	0.400
	(0.118)	(0.018)	(0.002)	(0.001)	$(3.63{\times}10^{-4})$	$(1.70{\times}10^{-4})$	$(1.57{\times}10^{-4})$	$(1.43{\times}10^{-4})$
β_1	−0.860	−1.099	−1.494	−1.503	−1.515	−1.500	−1.496	−1.497
	(0.860)	(0.644)	(0.093)	(0.056)	(0.038)	(0.011)	(0.011)	(0.009)
β_2	2.049	1.139	1.058	1.037	0.999	1.009	1.010	1.010
	(2.254)	(0.851)	(0.211)	(0.128)	(0.083)	(0.014)	(0.013)	(0.012)
β_3	−2.485	−3.091	−2.606	−2.570	−2.503	−2.507	−2.510	−2.510
	(2.314)	(1.214)	(0.342)	(0.194)	(0.113)	(0.021)	(0.020)	(0.018)
Peak	7	10	10	10	10	10	10	10
Take-off	—	5	5	5	5	5	5	5
Saturation	12	13	13	13	13	13	13	13
	Bayesian estimation: non-informative prior							
p	1.21	—	0.010	0.010	0.010	0.010	0.010	0.010
	(50.56)		$(1.10{\times}10^{-4})$	$(4.36{\times}10^{-5})$	$(2.42{\times}10^{-5})$	$(1.16{\times}10^{-5})$	$(1.15{\times}10^{-5})$	$(1.01{\times}10^{-5})$
q	−2.763	—	0.400	0.400	0.400	0.400	0.400	0.400
	(51.364)		(0.002)	(0.001)	$(3.44{\times}10^{-4})$	$(1.67{\times}10^{-4})$	$(1.56{\times}10^{-4})$	$(1.40{\times}10^{-4})$
β_1	1.531	—	−1.520	−1.517	−1.521	−1.500	−1.497	−1.496
	(55.183)		(0.119)	(0.055)	(0.035)	(0.011)	(0.010)	(0.009)
β_2	−0.293	—	0.988	1.003	0.984	1.009	1.010	1.009
	(55.894)		(0.291)	(0.135)	(0.075)	(0.014)	(0.013)	(0.012)
β_3	0.152	—	−2.502	−2.519	−2.484	−2.507	−2.51	−2.509
	(51.259)		(0.502)	(0.204)	(0.103)	(0.021)	(0.020)	(0.018)
Peak	1	—	8	10	10	10	10	10
Take-off	—	—	5	5	5	5	5	5
Saturation	—	—	13	13	13	13	13	13
	NLS estimation							
p	—	0.010	0.010	0.010	0.010	0.010	0.010	0.010
		—	$(2.16{\times}10^{-5})$	$(1.45{\times}10^{-5})$	$(1.21{\times}10^{-5})$	$(9.06{\times}10^{-6})$	$(9.95{\times}10^{-6})$	$(9.25{\times}10^{-6})$
q	—	0.401	0.400	0.400	0.400	0.400	0.400	0.400
		—	(3.4510^{-4})	(2.3610^{-4})	(1.7210^{-4})	(1.2910^{-4})	(1.3610^{-4})	(1.2610^{-4})
β_1	—	−0.941	−1.544	−1.543	−1.577	−1.501	−1.487	−1.488
		—	(0.079)	(0.07)	(0.071)	(0.033)	(0.035)	(0.033)
β_2	—	1.065	1.022	1.019	0.989	1.008	1.009	1.009
		—	(0.048)	(0.042)	(0.038)	(0.011)	(0.012)	(0.011)
β_3	—	−3.169	−2.615	−2.589	−2.480	−2.513	−2.519	−2.518
		—	(0.168)	(0.126)	(0.106)	(0.033)	(0.035)	(0.033)
Peak	—	10	10	10	10	10	10	10
Take-off	—	5	5	5	5	5	5	5
Saturation	—	13	13	13	13	13	13	13
	ML estimation							
p	—	0.010	0.009	0.010	0.010	0.010	0.010	0.010
		(0.001)	$(3.40{\times}10^{-4})$	$(3.05{\times}10^{-4})$	$(2.85{\times}10^{-4})$	$(2.50{\times}10^{-4})$	$(2.28{\times}10^{-4})$	$(2.30{\times}10^{-4})$
q	—	0.399	0.408	0.396	0.400	0.400	0.401	0.401
		(0.022)	(0.007)	(0.006)	(0.005)	(0.004)	(0.003)	(0.003)
β_1	—	−2.496	−1.114	−0.600	−1.618	−1.732	−0.769	−1.143
		(2.564)	(0.466)	(0.449)	(0.418)	(0.290)	(0.263)	(0.262)
β_2	—	0.734	0.309	0.005	0.630	−0.318	1.993	1.628
		(2.061)	(1.064)	(1.001)	(0.865)	(0.323)	(0.315)	(0.309)
β_3	—	−0.642	0.839	0.697	−1.973	−0.117	−2.672	−2.907
		(3.588)	(2.038)	(1.648)	(1.332)	(0.497)	(0.454)	(0.451)
Peak	—	8	10	9	10	9	9	9
Take-off	—	—	–	5	5	5	5	5
Saturation	—	12	13	13	13	13	13	13

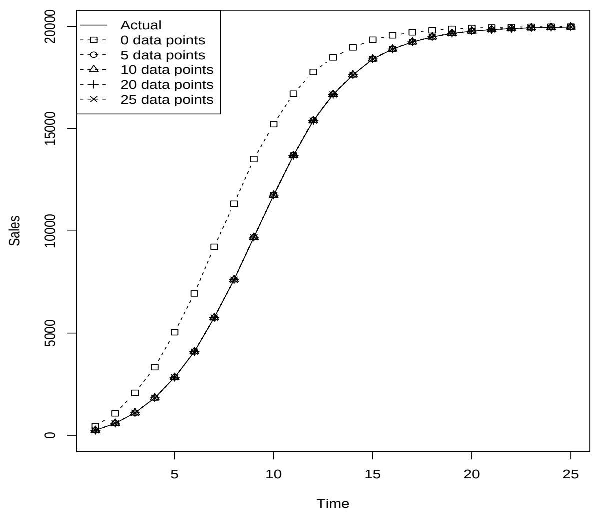
Notes: The models are fitted on each subsample up to t. Informative priors use information from previous markets, while non-informative priors are centered at zero with high variance. NLS and MLE include appropriate parameter restrictions. Standard errors are given in parentheses (standard deviation of chains for Bayesian methods and asymptotic for frequentist). "—" represents unavailable estimation results.

Table B.3: Bayesian Scaled-Beta2 and Inverse Gamma estimation without estimating the short-run market.

	$t=0$	$t=5$	$t=8$	$t=9$	$t=10$	$t=15$	$t=20$	$t=25$
				Scaled-Beta2				
p	0.018	0.010	0.010	0.010	0.010	0.010	0.010	0.010
	(0.008)	(4.82×10^{-4})	(1.10×10^{-4})	(5.32×10^{-5})	(2.66×10^{-5})	(1.16×10^{-5})	(1.16×10^{-5})	(1.00×10^{-5})
q	0.417	0.401	0.400	0.400	0.400	0.400	0.400	0.400
	(0.079)	(0.012)	(0.002)	(0.001)	(3.72×10^{-4})	(1.66×10^{-4})	(1.60×10^{-4})	(1.38×10^{-4})
β_1	-0.800	-0.888	-1.488	-1.510	-1.549	-1.497	-1.485	-1.486
	(0.589)	(0.812)	(0.279)	(0.206)	(0.141)	(0.041)	(0.042)	(0.037)
β_2	1.926	1.081	1.048	1.036	1.000	1.008	1.010	1.009
	(1.304)	(0.560)	(0.166)	(0.122)	(0.075)	(0.014)	(0.014)	(0.012)
β_3	-2.473	-3.056	-2.679	-2.636	-2.509	-2.514	-2.521	-2.518
	(1.541)	(1.47)	(0.535)	(0.367)	(0.212)	(0.043)	(0.042)	(0.037)
				Inverse Gamma				
p	0.019	0.010	0.010	0.010	0.010	0.010	0.010	0.010
	(0.013)	(1.05×10^{-4})	(3.27×10^{-5})	(2.08×10^{-5})	(1.56×10^{-5})	(1.01×10^{-5})	(1.06×10^{-5})	(9.91×10^{-6})
q	0.413	0.401	0.400	0.400	0.400	0.400	0.400	0.400
	(0.126)	(0.003)	(0.001)	(3.33×10^{-4})	(2.19×10^{-4})	(1.45×10^{-4})	(1.45×10^{-4})	(1.34×10^{-4})
β_1	-0.832	-1.358	-1.510	-1.510	-1.517	-1.500	-1.496	-1.497
	(1.072)	(0.245)	(0.032)	(0.025)	(0.024)	(0.009)	(0.009)	(0.009)
β_2	1.995	1.065	1.026	1.020	0.993	1.009	1.010	1.009
	(2.081)	(0.245)	(0.074)	(0.058)	(0.051)	(0.012)	(0.013)	(0.012)
β_3	-2.487	-2.840	-2.560	-2.545	-2.495	-2.506	-2.510	-2.509
	(3.050)	(0.365)	(0.128)	(0.087)	(0.070)	(0.018)	(0.019)	(0.018)

Notes: The models are fitted on each subsample up to t. Scaled-Beta2 has $a_0 = b_0 = 1$ and $\kappa_0 = 2,000$. The Inverse Gamma prior has both parameters equal to 0.001.

Figure B.3: Cumulative sales forecasts without estimating the short-run market.



Notes: Forecasts from Bayesian estimation with informative priors in the simulated data set. Parameter point estimates are used to compute full forecasts.

C The Scaled-Beta2 distribution

The beta distribution of the second kind, or *Beta2* distribution, can be obtained as the odds ratio from a beta distribution with parameters α and β. The *Scaled-Beta2* distribution can then be obtained using scaled odds.

Applying the univariate change of variables theorem to the following setting: let $X \sim \text{Beta}(\alpha, \beta)$ and $Y = g(X) = \kappa X/(1 - X)$, where κ is a scaling factor. Then, the absolute value of the Jacobian transformation is given by $\kappa/(Y + \kappa)^2$, and combining with the Beta distribution density function

$$f_X(X) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} X^{\alpha-1}(1 - X)^{\beta-1}$$

yields

$$\begin{aligned} f_Y(y) &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \left(\frac{y}{y + \kappa}\right)^{\alpha-1} \left(1 - \frac{y}{y + \kappa}\right)^{\beta-1} \frac{\kappa}{(y + \kappa)^2} \\ &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{1}{\kappa} \frac{(Y/\kappa)^{\alpha-1}}{(Y/\kappa + 1)^{\alpha+\beta}}\,, \end{aligned} \tag{C.1}$$

which is the probability density function for a Scaled-Beta2 distribution with parameters κ, α and β.

D Robustness checks

Table D.1: Bayesian estimation for different a priori variances.

	$\kappa = 2000$	$\kappa = \sigma_S^2$	$\kappa = 2\sigma_S^2$	$\kappa = 3\sigma_S^2$	$\kappa = 4\sigma_S^2$	$\kappa = 5\sigma_S^2$
p	0.010	0.010	0.010	0.010	0.010	0.010
	(1.16×10^{-5})	(1.17×10^{-5})	(1.18×10^{-5})	(1.19×10^{-5})	(1.15×10^{-5})	(1.18×10^{-5})
q	0.400	0.400	0.400	0.400	0.400	0.400
	(1.72×10^{-4})	(1.72×10^{-4})	(1.74×10^{-4})	(1.73×10^{-4})	(1.69×10^{-4})	(1.72×10^{-4})
m	19995.18	19995.225	19995.377	19995.215	19995.178	19995.180
	(5.815)	(5.955)	(5.611)	(5.845)	(5.793)	(5.815)
β_1	−1.487	−1.497	−1.497	−1.497	−1.497	−1.497
	(0.038)	(0.010)	(0.009)	(0.009)	(0.009)	(0.009)
β_2	1.009	1.009	1.009	1.009	1.010	1.009
	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)	(0.012)
β_3	−2.520	−2.510	−2.509	−2.510	−2.510	−2.509
	(0.037)	(0.018)	(0.018)	(0.018)	(0.019)	(0.019)

Notes: Bayesian estimation with an informative prior and differing prior variance. Standard errors are given in parentheses.

Table D.2: Mean squared errors (MSE) and mean absolute errors (MAE) for different a priori variances.

	$\kappa = 2000$	$\kappa = \sigma_S^2$	$\kappa = 2\sigma_S^2$	$\kappa = 3\sigma_S^2$	$\kappa = 4\sigma_S^2$	$\kappa = 5\sigma_S^2$
Noncumulative sales						
MSE	0.99	0.99	0.99	0.99	0.99	0.99
MAE	0.80	0.80	0.80	0.80	0.80	0.80
Cumulative sales						
MSE	1.52	1.64	1.45	1.67	1.59	1.62
MAE	1.06	1.10	1.05	1.11	1.09	1.10

Notes: Bayesian estimation with informative priors and differing prior variances.

Table D.3: Bayesian estimation for the air conditioner application with a misspecified short-run potential market.

	$m = 18{,}475$	$m = 15{,}000$	$m = 30{,}000$	$m = 50{,}000$
p	0.005	0.005	0.007	0.005
	(0.002)	(0.003)	(0.003)	(0.002)
q	0.354	0.420	0.222	0.170
	(0.020)	(0.042)	(0.020)	(0.023)
β_1	−1.072	−0.937	−1.089	−1.008
	(0.525)	(0.716)	(0.711)	(0.827)
β_2	0.539	0.440	0.771	0.824
	(0.243)	(0.364)	(0.374)	(0.469)

Notes: The first column shows the correct short-run potential market. Standard errors are given in parentheses.